Interacting growth processes and invariant percolation

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April 15, 2013

Abstract

The aim of this paper is to underline the relation between reversible growth processes and invariant percolation. We present two models of interacting branching random walks (BRWs), truncated BRWs and competing BRWs, where survival of the growth process can be formulated as the existence of an infinite cluster in an invariant percolation on a tree. Our approach is fairly conceptual and allows generalizations to a wider set of "reversible" growth processes.

KEYWORDS: survival, interacting branching random walk, invariant percolation, unimodular random networks

AMS 2000 MATHEMATICS SUBJECT CLASSIFICATION: 60J10, 60J80, 05C80

1 Introduction

In this note we discuss two different interacting branching random walks (BRWs) in discrete time. In the first model, called BRW_N, only a finite number N of particles are allowed per site. A natural question is whether the process BRW_N may survive with positive probability. Partially answers to this question were given by Zucca [27]. We complete these results for symmetric BRWs on Cayley graphs in Theorem 1.1: BRW_N survives with positive probability for sufficiently large N.

The second model concerns competing BRWs. Suppose there are two different types or species of particles: invasive and non-invasive particles. The invasive particles behave like particles in a usual BRW and are not influenced by the non-invasive particles. These latter however die once they share a site with an invasive particle. We prove, see Theorem 1.2, that for weakly surviving (or transient) BRWs on Cayley graphs both processes may coexist with positive probability.

Our proofs are based on a connection between the survival of reversible growth processes and the existence of infinite cluster in percolation on trees. This connection was used previously by Schramm [23] and Benjamini and Mueller [7]. In the first reference BRWs are used to study connectivity properties of Bernoulli percolation on non-amenable Cayley graphs. Benjamini and Mueller [7] use results on invariant percolation (on trees) to study BRWs on unimodular random graphs.

In general the study of interacting growth processes or particles systems is challenging and a general treatment seems to be out of reach (at least at the moment). The case-by-case study

often involves a high amount of technical effort. The approach given here is more conceptual using *soft proofs*. While we concentrate on two concrete examples in this paper we want to underline that our approach is fairly general and only relies on two steps: the formulation of the process as a unimodular random network and the control of the marginal of the corresponding invariant percolation.

1.1 Motivation

Besides highlighting the connection between growth processes and percolation there are several other motivations for the underlying work. One of these motivations is to propose models for spatial interaction and competition of growth processes. One of the earliest and simplest models of growth processes is the Galton-Watson branching process where particles branch independently of the history of the process. However, this may not be very realistic when there is competition for limited resources such as space and food in the habitat. A considerable effort was made to introduce dependence in the sense that the individual reproduction may be influenced by the history of the population. While many of these models consider dependence only on the total population size, we refer to Kersting [21] and the paper referring to it for a mathematical introduction, the study of models with local interactions is perhaps even more challenging. Models in this direction are for example contact processes or restrained BRWs. Here, particles breed depending on the local configurations of the particles and one is interested in extinction, equilibrium and explosion of the process. We refer to Bertacchi *et al.* [10] and references therein. A natural way to model local dependencies between particles is also the truncated BRW_N introduced in Zucca [27].

As a byproduct of our approach we can also control the following processes. In a first model dependencies are not only between particles of the same generation but between particles of different generations. Suppose that each vertex has a finite amount of resources that allow at most N particles to branch; once the resources are used any other particle visiting this site will die without producing any offspring. For this model an analogue result of Theorem 1.1 holds for the weakly surviving regime, see Section 3. Another model of annihilating BRW that can be treated is the process where the probability that two particles, meeting at a same vertex, annihilate each other is a function of their distance in the family tree. For instance, particles annihilate each other if and only if their distance in the family tree is larger than some constant M. Despite the non-monotonicity of the model one can prove that for M large enough the process survives in the weakly surviving regime.

Our second model describes two species competing for resources and studies whether the weaker (or non-invasive) species has a chance of survival. Models for competing spatial growth attracted a lot of attention in the last decades. Perhaps the most common models in the probability community are the voter model, the Richardson model, and mixtures of these two. We refer to Hägström and Pemantle [18], Kordzakhia and Lalley [22], and Blair-Stahn [11] for an introduction and more references. Let us note that most of the results are restricted to $\mathbb{Z}^d$ and make strong use of a connection with first passage percolation. Our model constitutes, to our

knowledge, one of the first models beyond $\mathbb{Z}^d$ and is more realistic for models where the space of possible habitat grows exponentially (is expanding) in time. In particular, this is relevant for models at the early stage of competing populations. Moreover, it provides a stochastic model for the so called dominance displacement competition, we refer to Amarasekare [3] for more details and references. In these kind of models superior competitors can displace inferior competitors but not vice-versa. However, the inferior competitors can establish "patches or niches" where the superior competitor do not colonize. This latter phenomomen is highlighted by our theoretical result: as long as the superior competitor does not colonize the whole space the free patches are large enough to allow the inferior competitor to survive.

A somehow completely different motivation originates from a structure theoretic question. Classification of groups in terms of the behavior of random processes attracted a lot of attention. In particular, a consequence of Gromov's famous theorem on groups of polynomial growth is that a finitely-generated group admits a recurrent random walk if and only if it contains a finite-index subgroup isomorphic with $\mathbb{Z}$ or $\mathbb{Z}^2$, *e.g.*, see Chapter 3 in Woess [26]. Kesten's criterion for amenability says that a finitely-generated group is amenable if and only if the spectral radius for any (or some) symmetric random walk is equal to 1. This phenomenon is also underlined by phase transitions on non-amenable graphs whose study underwent a rapid development, *e.g.*, see Lyons [25]. Moreover, Benjamini [4] proposed a deterministic competition model that admits coexistence on hyperbolic groups but not on $\mathbb{Z}^d$. A motivation for this is to find a stochastic process (or a system of interacting processes) that shows an additional phase-transition on (one-ended) hyperbolic groups compared to non-hyperbolic groups. Theorem 1.2 shows that coexistence of competing BRWs, at least in the weakly surviving regime, occurs regardless of the hyperbolicity of the underlying graph.

1.2 Structure of the paper

We formulate our models and corresponding main results, Theorem 1.1 and Theorem 1.2, in the rest of this section. In Section 2 we introduce the notations and basic results of random unimodular networks (URNs) and present two preliminary results on percolation of URNs, Lemma 2.3 and Theorem 2.4. Section 3 contains the proof of Theorem 1.1 and Section 4 the one of Theorem 1.2.

1.3 Branching random walks

The definition of a branching random walk (BRW) requires a probability distribution, $\mu = (\mu_k)_{k\geq 0}$, describing the branching and a transition kernel, $P = (p(x,y))_{x,y\in G}$ describing the movement of the particles on some underlying discrete space G. The BRW starts at some initial position o with one particle and then at each (discrete) time step each particle splits into k particles with probability μ_k and each of the resulting particles moves according to the transition kernel P. Both splitting and movement of a particle at time n are independent of the previous history of the process and the behavior of the other particles at time n.

The mean number of offspring is denoted by $m = \sum_k k\mu_k$ and we will always assume that the corresponding Galton-Watson process is supercritical, *i.e.*, $m > 1$. Furthermore, we assume that P is the transition kernel of an irreducible random walk. Therefore, the spectral radius $\rho = \rho(P) = \limsup_{n\to\infty}(p^{(n)}(x,x))^{1/n}$, $x \in G$, of the underlying random walk is well defined.

There is an alternative description of BRWs that uses the concept of tree-indexed random walks introduced in [8]. Let $(\mathbb{T}, \mathbf{r})$ be a rooted infinite tree. Denote by v the vertices of $\mathbb{T}$ and let $|v|$ be the (graph) distance from v to the root $\mathbf{r}$. For any vertex v denote v^- the unique predecessor of v, *i.e.*, $v^- \sim v$ and $|v^-| = |v| - 1$. We often identify a Graph G with its vertex set and denote the latter also by G. Moreover, the Cayley graph of a finitely generated group G with respect to some generating set S will also be denoted by G.

Let (G, o) be a rooted graph. The tree-indexed process $(S_v)_{v\in\mathbb{T}}$ is defined inductively such that $S_{\mathbf{r}} = o$ and for vertices $x, y \in G$ we have

$$\mathbb{P}\left(S_v = x | S_{v^-} = y, \{S_w :\ w \notin \{v, v^-\}, |w| \leq n\}\right) = \mathbb{P}(S_v = x | S_{v^-} = y) = p(x, y).$$

A tree-indexed random walk becomes a BRW if the underlying tree $\mathbb{T}$ is a realization of a Galton-Watson process. We call $\mathbb{T}$ the family tree and G the base graph of the BRW.

In the case when G is a Cayley graph (or finitely generated group) the BRW can be described as a labeled Galton-Watson tree. Let G be a finitely generated group with group identity o and write the group operations multiplicatively. Let q be a symmetric probability measure on a finite symmetric generating set of G. The corresponding random walk on G is the Markov chain with state space G and transition probabilities $p(x, y) = q(x^{-1}y)$ for $x, y \in G$. Equivalently, the random walk (starting in x) can be described as

$$S_n = xX_1 \cdots X_n, \quad n \geq 1,$$

where the X_i are i.i.d. random variables with distribution q. Now, label the edges of $\mathbb{T}$ with i.i.d. random variables X_v with distribution q; the random variable X_v is the label of the edge (v^-, v). Define $S_v = o \cdot \prod_i X_{v_i}$ where $\langle v_0 = \mathbf{r}, v_1, \ldots, v_n = v\rangle$ is the unique geodesic from $\mathbf{r}$ to v at level n.

A BRW is said to **survive strongly (or locally)** if every vertex is visited infinitely many times with positive probability and to **survive weakly** if the process survives with positive probability but every finite subset is eventually free of particles. In formulæ:

$$\text{strong survival} \Leftrightarrow \forall x \in G :\ \mathbb{P}(|\{v : S_v = x\}| = \infty) > 0$$

$$\text{weak survival} \Leftrightarrow \mathbb{P}(|\mathbb{T}| = \infty) > 0 \text{ and } \forall x \in G :\ \mathbb{P}(\{v : S_v = x\}| = \infty) = 0.$$

Important to note that several authors speak sometimes of **transience** and **recurrence** of BRWs instead of weak and strong survival. As a consequence of the classification of recurrent groups and Kesten's amenability criterion we have that a BRW on a Cayley graph survives locally if and only if $m\rho(P) \leq 1$, see also [15] for an alternative proof.

We make the following standing assumptions on the underlying probability measures.

Assumption 1.1.

- *The underlying Galton-Watson process is supercritical,* i.e., $m = \sum_k k\mu_k > 1$*, and the offspring distribution* μ *is of finite support,* i.e., *there exists some* d *such that* $\sum_{k=0}^{d-1} \mu_k = 1$*. Furthermore, we assume that* $\mu_1 > 0$*.*
- *Let* G *be a finitely generated group with symmetric finite generating set* S*. The distribution* q *of the random walk on* G *is symmetric and such that* $supp(q) = S$ *and* $q(e) > 0$*.*

Remark 1.1. While the assumptions on supercriticality of the Galton-Watson process and symmetry and irreducibility of the random walk are crucial, the other assumptions are made for sake of a better presentation and to avoid periodic subtleties. In particular, the assumption on finite support of the offspring distribution can be removed from the BRW_N and the non-invasive process by adding an additional coupling of the Galton-Watson process with a "truncated" version. The assumption that the genealogy of the invasive BRW is a Galton-Watson process with finite support is not needed anywhere.

1.4 Truncated branching random walk

Branching random walks may be used to describe the evolution of a population or particle system at early stage with no restrictions on resources. In order to refine the model one might introduce a limit of particle at each site: for some $N \in \mathbb{N}$ at most N particles are allowed at a same site at the same time. While most of the existing models describing variants of this models are in continuous time, we prefer a description in discrete time since our proof technique is more suitable to this setting. Let $N \in \mathbb{N}$ and V be a finite set and denote by $C(V, N)$ a random variable that chooses uniformly N elements from the set V with the convention that if $N < |V|$ then all $|V|$ elements are chosen. All random variables of the kind $C(V, N)$ that will appear are supposed to be independent of everything else.

We define an auxiliary process: let S_v^{aux} be a BRW with offspring distribution μ and transition kernel P and denote by $\mathbb{T}^{aux}$ the corresponding family tree. For every $x \in X$ and $n \in \mathbb{N}$ we denote by $V_{n,x} = \{v : \ |v| = n, S_v^{aux} = x\}$ the particles at generation n that are in position x. We add a special state $\dagger$ to the state space and define the process BRW_N on $G \cup \{\dagger\}$ as

$$S_v^N = \begin{cases} S_v^{aux} & \text{if } v \in C(V_{n,S_v^{aux}}, N) \\ \dagger & \text{otherwise,} \end{cases} \quad \text{for all } v \in \mathbb{T}^{aux}. \tag{1}$$

The state $\dagger$ induces a site percolation on the family tree $\mathbb{T}^{aux}$ in the following way: declare a vertex v closed if $S_v^{aux} = \dagger$ and open otherwise. Configurations of this percolation are denoted by $\eta_\dagger$, where $\eta_\dagger(v) = 1$ corresponds to the fact that the site v is open and $\eta_\dagger(v) = 0$ to the fact that site v is closed. We denote by $\mathbb{T}_\mathbf{r}$ the connected component containing the root.

The truncated process BRW_N can therefore be denoted by $(S_v^N)_{v \in \mathbb{T}_r}$ and the question of survival of the truncated process is equivalent to whether $|\mathbb{T}_\mathbf{r}| = \infty$. It is easy to see that survival is a monotone property: the process survives for $N_2 > N_1$ if it survives for N_1. The following results asserts that there exists a non-trivial phase transition.

Theorem 1.1. *Let P be the transition kernel of a symmetric irreducible random walk on an infinite finitely generated group G and let μ be an offspring distribution with $m > 1$. Then, there exists a critical value $N_c = N_c(\mu, P) < \infty$ such that if $N \leq N_c$ the process dies out a.s. and if $N > N_c$ the process survives with positive probability.*

Remark 1.2. Some of the results have be proved by Zucca [27]: the case $m > 1/\rho(P)$ was settled completely but the case $m \leq 1/\rho(P)$ was only proven for some BRWs on $\mathbb{Z}^d$ and on the homogeneous tree. Zucca's results are presented in a more general context, *e.g.*, quasi-transitivity of BRW, and treat some BRW with drift on $\mathbb{Z}^d$ that are not covered by our result. While his proof technique is different to ours, it is interesting to note that he uses as well a percolation argument for the case $m > 1/\rho(P)$; this time directed percolation on products of $\mathbb{N}$.

1.5 Competing branching random walks

We consider two competing BRWs that interact in the following way. One BRW is **i**nvasive, *i.e.,* the particles do not care about the other particles, and the second is **n**on-invasive in the sense that once a particle shares a site (at the same time) with an invasive particle it dies without having any offspring. The particles live on an infinite finitely generated group G and we note $(\mu_{\mathbf{i}}, P_{\mathbf{i}})$, $(\mu_{\mathbf{n}}, P_{\mathbf{n}})$ for their offspring distribution and transition kernels. Moreover, denote by $m_{\mathbf{i}}$ and $m_{\mathbf{n}}$ their mean number of offspring.

We give a formal definition of a slightly different process in the following. The branching distributions $\mu_{\mathbf{i}}$ and $\mu_{\mathbf{n}}$ give rise to two family trees $\mathbb{T}^{\mathbf{i}}$ and $\mathbb{T}^{\mathbf{n}}$. The non-invasive BRW will start in o and the invasive in some point $x \neq o$.

The invasive BRW $(S_v^{\mathbf{i}})_{v \in \mathbb{T}^{\mathbf{i}}}$ is defined as an ordinary BRW. In order to define the non-invasive BRW we first construct an intermediate version. Let S_v^{aux} be an ordinary BRW with $(\mu_{\mathbf{n}}, P_{\mathbf{n}})$ and introduce an additional state $\dagger$. The non-invasive BRW $G \cup \{\dagger\}$ is defined together with $(S_v^{\mathbf{i}})_{v \in \mathbb{T}^{\mathbf{i}}}$ on a joint probability space such that

$$S_v^{\mathbf{n}} = \begin{cases} S_v^{aux} & \text{if } S_w^{\mathbf{i}} \neq S_v^{aux}\ \forall w \in \mathbb{T}^{\mathbf{i}}_{|v|}, \\ \dagger & \text{otherwise,} \end{cases} \quad \text{for } v \in \mathbb{T}^{aux}. \tag{2}$$

We denote by $\mathbb{P} = \mathbb{P}_{o,x}$ the canonical probability measure describing both processes on a same probability space.

We introduce a percolation of the family tree $\mathbb{T}^{aux}$ by declaring a vertex $v \in \mathbb{T}^{aux}$ closed if and only if $S_v^{aux} = \dagger$. Configuration of this percolation are denoted by $\eta_\dagger$. We denote by $\mathbb{T}_{\mathbf{r}}^{\mathbf{n}}$ the connected component of $\mathbb{T}^{aux}$ containing the root $\mathbf{r}$.

We say that there is **coexistence** if with positive probability both processes survive, *i.e.,* $\mathbb{P}_{o,x}(|\mathbb{T}_{\mathbf{r}}^{\mathbf{n}}| = \infty, |\mathbb{T}^{\mathbf{i}}| = \infty) > 0$. Using the assumptions $\mu_0 > 1$ and $q(e) > 0$ together with the Markov property one sees that if $\mathbb{P}_{o,x}(|\mathbb{T}_{\mathbf{r}}^{\mathbf{n}}| = \infty, |\mathbb{T}^{\mathbf{i}}| = \infty) > 0$ holds for some x then it holds for all $x \neq o$.

Theorem 1.2. *Let $P_{\mathbf{i}}$ and $P_{\mathbf{n}}$ transition kernels of random walks on a infinite finitely generated group G and let $\mu_{\mathbf{i}}$ and $\mu_{\mathbf{n}}$ satisfying Assumption 1.1. Then, there is coexistence of the invasive and the non-invasive process if $m_{\mathbf{i}}\rho_{\mathbf{i}} < 1$.*

Remark 1.3 (Strongly surviving regime). Theorem 1.2 states that there is always coexistence if the invasive BRW is weakly surviving. Since we assume the underlying random walk to be symmetric the result does not apply to BRW on $\mathbb{Z}_d$. This is because Kesten's amenability criterion implies that there is no (symmetric) weakly surviving BRW on amenable groups (including $\mathbb{Z}^d$). However, on $\mathbb{Z}^d$ one can show that there is no coexistence if $m_{\mathbf{i}} > m_{\mathbf{n}}$. This can be seen by proving a shape theorem using large deviation estimates of the underlying random walks. We refer to [14] where a shape theorem was established even in a random environment. However, this approach fails for groups beyond $\mathbb{Z}^d$ since large deviations for random walks on groups are up to now not sufficiently studied. Moreover, there is no reason why the shape of the particles should be a "convex set"; see also [19] and [12] for results on groups with infinitely many ends. Hence, one may ask in the flavor of Benjamini [4]: does coexistence in the strong surviving regime, $m_{\mathbf{i}}\rho_{\mathbf{i}} > 1$ and $m_{\mathbf{n}}\rho_{\mathbf{n}} > 1$, depend on the hyperbolicity of the base graph?

Remark 1.4 (Critical case). We can not treat the critical case, $m_i\rho_{\mathbf{i}} = 1$, in general since we do not know for which groups and walks the Green function $G(x, y|\rho_{\mathbf{i}}^{-1})$ decays exponentially in $d(x, y)$. However, this is true for finite range symmetric random walks on hyperbolic groups, see [16], and hence our methods also cover the critical case in this setting.

Remark 1.5. On groups with infinitely many ends we have that on the event of co-existence not every non-invasive particle has an offspring that will be killed. This is due to the fact that the invasive BRW leaves some neighborhoods of the boundary unvisited, see [19] and [12], where the non-invasive process may live in peace. The results in [24] strongly suggests that this is still true for Fuchsian groups and one is tempted to ask if this phenomenon holds true for general groups. Since the shape of a single BRW is connected to the question of co-existence for competing BRWs, see also Remark 1.3, an answer to this question seems to be related to the conjecture that the trace of a weakly surviving BRW has infinitely many ends, see [7].

2 Preliminaries

2.1 Unimodular random graphs

Unimodular random graphs (URGs) or stochastic homogeneous graphs have several motivations and origins. We concentrate in this note on the probabilistic point of view since it gives rise to the tools we are going to use. For more details on the probabilistic viewpoints we refer to [2], [5], [6] and to [20] for an introduction to the ergodic and measure theoretical origins.

One of our motivation to consider unimodular random graphs is the use of a *general* Mass-Transport Principle (MTP) which was established in [9] under the name of "Intrinsic Mass-Transport Principle". It was motivated by the fact that the Mass-Transport Principle is heavily used in percolation theory and therefore lifts many results on unimodular graphs to a more general class of graphs. In [2] a probability measure on rooted graphs is called unimodular if this general form of the MTP holds. Another motivation to consider URG is the fact that unimoduar random trees (URTs) can be seen as connected components in an invariant percolation on trees,

see [6, Theorem 4.2] or Theorem 2.2 in this paper.

Let us define URGs properly. We write (G, o) for a graph G with root o. A rooted graph (G, o) is isomorphic to (G', o') if there exists an isomorphism of G onto G' which takes o to o'. We denote by $\mathcal{G}_*$ the space of isomorphism classes of rooted graphs and write $[G, o]$ for the equivalence class that contains (G, o). The space $\mathcal{G}_*$ is equipped with a metric that is induced by the following distance between two rooted graphs (G, o) and (G', o'). Let α is the supremum of those $r > 0$ such that there exists some rooted isomorphism of the balls of radius $\lfloor r \rfloor$ (in graph distance) around the roots of G and G' and define $d((G, o), (G', o')) = 1/(1 + \alpha)$. In this metric $\mathcal{G}_*$ is separable and complete. In the same way one defines the space $\mathcal{G}_{**}$ of isomorphism classes of graphs with an ordered pair of distinguished vertices. A Borel probability measure ν on $\mathcal{G}_*$ is called unimodular if it obeys the Mass-Transport Principle: for all Borel function $f : \mathcal{G}_{**} \to [0, \infty]$, we have

$$\int \sum_{x \in V} f(G, o, x) d\nu([G, o]) = \int \sum_{x \in V} f(G, x, o) d\nu([G, o]). \tag{3}$$

Observe that this definition can be extended to labeled graphs or networks. A network is a graph $G = (V, E)$ together with a complete metric space Ξ and maps from E and V to Ξ. While the definition of the above equivalence classes for networks is straightforward, one has to adapt the metric between two networks as follows: the α is chosen as the supremum of those $r > 0$ such that there is some rooted isomorphism of the balls of radius $\lfloor r \rfloor$ around the roots of G and G' and such that each pair of corresponding labels has distance at most $1/r$. A probability measure on rooted labels networks is called unimodular if the to Equation (3) corresponding principle holds. Realizations of these measures or denoted as unimodular random networks (URN). Following the existing literature we use the notation (G, o) as well for networks and specify the labels and marks of a network only when it is necessary.

Let us illustrate this definition with the very important examples of Galton-Watson measures. Let $\mu = \{\mu_k\}_{k \in \mathbb{N}}$ be a probability distribution on the integers. The Galton-Watson tree is defined inductively: start with one vertex, the root of the tree. Then, the number of offspring of each particle (vertex) is distributed according to μ. Edges are between vertices and their offspring. We denote by **GW** the corresponding measure on the space of rooted trees. In this construction the root clearly plays a special role. In the unimodular Galton-Watson measure (**UGW**) the root has a degree biased distribution: the probability that the root has degree $k+1$ is proportional to $\frac{\mu_k}{k+1}$. In cases where we use the **UGW** measure instead of the standard **GW** measure to define the family tree of the BRW we denote the BRW by UBRW.

Lemma 2.1 (Lemma 4.1, [6])**.** *Suppose that ν is a unimodular measure on rooted networks. Let ϕ be a measurable map on rooted networks that takes each network to an element of the mark space. Define Φ to be the map on rooted networks that takes a network (G, o) to another network on the same underlying graph, but replaces the mark at each vertex $x \in G$ by $\phi(G, x)$. Then, the push forward measure $\Phi_* \nu$ is also unimodular. If instead we add an additional coordinate to each mark by an i.i.d. mark, then the resulting measure is again unimodular.*

The last statement implies immediately that an i.i.d. labeled UGW-tree is again a unimodular random network. Hence, using the definition of the UBRW as a tree-indexed random walk we can see the UBRW as an i.i.d. labeled URT. We denote by $\mathbf{UGW}_q$ for this measures on unimodular random networks.

Theorem 2.2 (Theorem 4.2, [6])**.** *Let ν be a probability measure on rooted networks whose underlying graphs are trees of degree at most d. Then ν is unimodular iff ν is the law of the open component of the root in a labeled percolation on a d-regular tree whose law is invariant under all automorphisms of the tree.*

Let ν be a unimodular measure on rooted networks (G, o) and suppose that the mark space Ξ contains a particular mark $\dagger$. This special mark induces a natural percolation on the rooted network: a vertex is closed if its mark equals to $\dagger$ and open otherwise.

Lemma 2.3. *Let ν be a unimodular measure on rooted networks. Let $\dagger$ be a particular element of the mark space that induces a percolation. Denote by (C, o) the connected (labeled) component containing the origin and denote by $\nu_\dagger$ its corresponding measure. Then, the measure $\nu_\dagger$ is again a unimodular measure on rooted networks.*

Proof. The proof is a check of the Mass-Transport Principle (3). We write $\{(G, o) \rightsquigarrow (C, o)\}$ or just $\{G \rightsquigarrow C\}$ for the set of rooted networks where the induced percolation leads to (C, o) as the connected component containing the origin. For any positive borel function $f : \mathcal{G}_{**} \to [0, \infty]$ define its "restrictions"

$$g_{|C}(G, x, y) = \begin{cases} f(C, x, y), & \text{if } x, y \in C, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \int_{(C,o)} \sum_{x \in C} f(C, o, x) d\nu_\dagger[C, o] &= \int_{(C,o)} \sum_{x \in C} f(C, o, x) d\nu[G \rightsquigarrow C] \\ &= \int_{(C,o)} \int_{\tilde{G} \in \{G \rightsquigarrow C\}} \sum_{x \in \tilde{G}} g_{|C}(\tilde{G}, o, x) \frac{d\nu[\tilde{G}]}{\nu[\{G \rightsquigarrow C\}]} d\nu[\{G \rightsquigarrow C\}] \end{aligned}$$

where

$$\int_{\tilde{G} \in \{G \rightsquigarrow C\}} \sum_{x \in \tilde{G}} g_{|C}(\tilde{G}, o, x) \frac{d\nu[\tilde{G}]}{\nu[\{G \rightsquigarrow C\}]} = \int_{\tilde{G}} \sum_{x \in \tilde{G}} \mathbf{1}\{\tilde{G} \in \{G \rightsquigarrow C\}\} \frac{g_{|C}(\tilde{G}, o, x)}{\nu[\{G \rightsquigarrow C\}]} d\nu[\tilde{G}].$$

Hence, unimodularity of $\nu_\dagger$ follows by unimodularity of ν applied to the functions

$$h(\tilde{G}, x, y) = \mathbf{1}\{\tilde{G} \in \{G \rightsquigarrow C\}\} \frac{g_{|C}(\tilde{G}, o, x)}{\nu[\{G \rightsquigarrow C\}]}.$$

□

In percolation theory one is naturally interested in the existence of infinite clusters. There are results by Adams and Lyons [1] and Häggström [17] that state that for invariant percolation on homogeneous trees a sufficiently high marginal guarantees (with positive probability) the existence of infinite clusters. We generalize this result to "invariant percolation" on supercritical Galton-Watson trees. This is done by adapting the proof of Theorem 1.6 in [17].

Theorem 2.4. *Let* $\boldsymbol{UGW}$ *be a supercritical unimodular Galton-Watson measure of maximal degree* d*. Then, there exists some* $c_{\boldsymbol{UGW}} < 1$ *such that for any unimodular labeling and any particular element* $\dagger$ *of the mark space the induced percolation* $\boldsymbol{UGW}_\dagger$ *assigns positive probability to the existence of infinite clusters if the marginal is greater than* $c_{\boldsymbol{UGW}}$*.*

Proof. Theorem 2.2 and Lemma 2.3 imply that $\mathbf{UGW}_\dagger$ defines an invariant (site) percolation of the homogeneous tree of degree d. As we have to treat two interlaced percolations, we denote $\eta_{\mathbf{UGW}}$ for configurations of the percolation induced by $\mathbf{UGW}$ and $\eta_\dagger$ for configurations induced by $\mathbf{UGW}_\dagger$. From the definition of $\mathbf{UGW}_\dagger$ we have that components that are connected in $\eta_\dagger$ are also connected in $\eta_{\mathbf{UGW}}$. For any vertex v in $\mathbb{T}_d$ and a given configuration η we write $C_\eta(v)$ for the connected component containing e in η. We denote by ∂C the outer (vertex) boundary of a vertex set C. We can now adapt the proof of Theorem 1.6 in [17]. Given a configuration $\eta_\dagger$ we define a function $\psi_{\eta_\dagger}$ on the vertex set of $\mathbb{T}_d$. For a vertex v denote by $v_1, \dots, v_d$ its adjacent vertices in $\mathbb{T}^d$ and let

$$\psi(v) = \begin{cases} 1 & \text{if } \eta_\dagger(v) = 1 \text{ and } |C_{\eta_\dagger}(v)| = \infty, \\ 1 & \text{if } \eta_\dagger(v) = 1, |C_{\eta_\dagger}(v)| < \infty \text{ and } \frac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} \geq K, \\ 0 & \text{if } \eta_\dagger(v) = 1, |C_{\eta_\dagger}(v)| < \infty \text{ and } \frac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} < K, \\ 1 + \sum_{i=1}^d f(v_i) & \text{if } \eta_\dagger(v) = 0, \end{cases} \tag{4}$$

where

$$f(w) = \begin{cases} \frac{|C_{\eta_\dagger}(w)|}{|\partial C_{\eta_\dagger}(w)|} & \text{if } |C_{\eta_\dagger}(w)| < \infty \text{ and } \frac{|C_{\eta_\dagger}(w)|}{|\partial C_{\eta_\dagger}(w)|} < K, \\ 0 & \text{otherwise,} \end{cases}$$

for some positive constant K to be chosen later. We can now, as in the proof Theorem 1.6 in [17], interpret ψ as a distribution of mass over the vertices. Originally every vertex has mass 1. For vertices v in an infinite cluster or vertices v in finite clusters with $\frac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} \geq K$ the mass in v remains unchanged. If v is in a finite cluster such that $\frac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} < K$ then v distributes its mass equally to the closed vertices incident to $C_{\eta_\dagger}(v)$. If $\eta_\dagger(v) = 0$, then v receives additional mass from the distributing vertices. For two vertices v and w we write $\triangle\psi(v,w)$ for the flow of mass from v to w (using the above interpretation) and obtain

$$\psi(v) = 1 + \sum_w \triangle\psi(w,v).$$

Consider random configurations $X_{\mathbf{UGW}}$ and $X_\dagger$ that are distributed according to $\mathbf{UGW}$ and $\mathbf{UGW}_\dagger$. Since $\mathbf{UGW}_\dagger$ is unimodular we have that for any pair of vertices v and w that

$$\mathbb{E}_{\mathbf{UGW}_\dagger}[\triangle\psi(v,w)] = 0.$$

Since ψ is bounded we obtain that

$$\mathbb{E}_{\mathbf{UGW}_\dagger}[\psi(v)] = 1 + \sum_w \mathbb{E}_{\mathbf{UGW}_\dagger}[\triangle\psi(w,v)] = 1.$$

Using this together with the definition of ψ in Equation (4) we obtain

$$\begin{aligned}\mathbf{UGW}_\dagger(X_\dagger(v)=1, |C_{X_\dagger}(v)|=\infty) \quad &\geq \quad 1 - \mathbf{UGW}_\dagger\left(X_\dagger(v)=1, |C_{X_\dagger}(v)|<\infty, \frac{C_{X_\dagger}(v)|}{|\partial C_{X_\dagger}(v)|} \geq K\right) \\ &\quad -(1+dK)\mathbf{UGW}_\dagger(X_\dagger(v)=0).\end{aligned}$$

In order to adjust the value of K recall the following. The anchored (vertex) isoperimetric constant for a graph G is defined as

$$\mathbf{i}(G,v) = \inf_{S \ni v} \frac{|\partial S|}{|S|},$$

where S ranges over all connected vertex sets containing a fixed vertex v. Note that $\mathbf{i}(G,e)$ does not depend on the choice of the edge v. Corollory 1.3 in [13] states that $\mathbf{i}(\mathbb{T},v)>0$ a.s. on the event that $\mathbb{T}$ is infinite. Now, since

$$\mathbf{UGW}_\dagger\left(X_\dagger(v)=1, |C_{X_\dagger}(v)|<\infty, \frac{C_{X_\dagger}(v)|}{|\partial C_{X_\dagger}(v)|} \geq K\right) \leq \mathbf{UGW}\left(\mathbf{i}(\mathbb{T})^{-1} > K, |C_{X_{\mathbf{UGW}}}(v)|=\infty\right),$$

we can choose K sufficiently large such that

$$\mathbf{UGW}_\dagger\left(X_\dagger(v)=1, |C_{X_\dagger}(v)|<\infty, \frac{C_{X_\dagger}(v)|}{|\partial C_{X_\dagger}(v)|} \geq K\right) < \mathbf{UGW}\left(|C_{X_{\mathbf{UGW}}}(v)|=\infty\right).$$

Eventually, there exists some constant $c>0$ such that

$$\mathbf{UGW}_\dagger(|C_\dagger(v)|=\infty \mid |C_{X_{\mathbf{UGW}}}(v)|=\infty) > c - \frac{(1+dK)\mathbf{UGW}_\dagger(X_\dagger(v)=0))}{\mathbf{UGW}(|C_{X_{\mathbf{UGW}}}(v)|=\infty)}.$$

Hence, choosing the marginal $\mathbf{UGW}_\dagger(X_\dagger(v)=1)$ sufficiently high assures that $\mathbf{UGW}_\dagger(|C_{X_\dagger}(v)|=\infty)>0$. □

Remark 2.1. An inspection of the proof above reveals that Theorem 2.4 holds true for unimodular measures on rooted networks whose underlying graphs are trees of bounded degree and that give positive weight to infinite networks such that almost all infinite realizations have positive anchored isoperimetric constant.

3 Truncated BRW - proof of Theorem 1.1

Since the case $m>1/\rho(P)$ was proven in [27] let us assume in the following that $m \leq 1/\rho(P)$. In the case $\mu_0=0$ the proof is essentially given in the proof of Theorem 3.1 in [7]. We give a concise proof using the results of [6] and Theorem 2.4. Moreover, we hope that the example of truncated BRW will serve as an introduction of our approach “interacting growth process and invariant percolation” and hence is useful for a better understanding of the proof of Theorem 1.2. Our approach consists of two steps: an adaptation of the model such that the family tree is a URT and the control of the marginal density of the corresponding invariant percolation.

3.1 Adapting the model

The aim is to identify an invariant percolation (or unimodular measure) of the family tree. Since the percolation induced by † is in not an invariant percolation, there is need for a reformulation of our problem. We will define a new process in a way such that vertices that were visited more than N times become "deadly" for all instances of times. In other words, if x is a vertex of the base graph such that $|\{v: S_v = x\}| > N$, then we set $S_v^{new} = \dagger$ for all v such that $S_v = x$. More formally, let $(\mathbb{T}, \mathbf{r})$ be the labeled UGW-tree (the BRW) and define

$$\phi(\mathbb{T}, \mathbf{r}) = \begin{cases} \dagger, & |\{v: S_v = x\}| > N, \\ \bullet, & |\{v: S_v = x\}| \leq N. \end{cases} \tag{5}$$

The corresponding push forward measure $\Phi_* \mathbf{UGW}_q$ is again unimodular, see Lemma 2.1.

3.2 Control of the marginal

The underlying BRW is supposed to be weakly surviving, *i.e.*, $P(|\{v : S_v = S_{\mathbf{r}}\}| < \infty) = 1$. Hence, we can apply Theorem 2.4 and choose N_u sufficiently large such that the marginal $\mathbb{P}(|\{v : S_v = S_{\mathbf{r}}\}| \leq N_u)$ is sufficiently high. This guarantees that with positive probability the cluster containing $\mathbf{r}$ is infinite. Hence, the truncated process BRW_N survives with positive probability for sufficiently large N. This yields, together with the monotonicity of the model, the existence of a critical value N_c given in Theorem 1.1.

4 Competing BRWs - proof of Theorem 1.2

We proceed in two steps as in the previous section.

4.1 Adapting the model

The family of the non-invasive process is in general not a URT. We invite the reader to the following informal description of the situation.

Let us start both processes in neighboring sites, then the offspring of the starting particles are very likely to be killed by those of the invasive process. However, if we consider some non-invasive particle very late in the genealogical process, then, given the fact that the particles exists (or is alive), one might expect that the invasive particles never have been very close to its parents. Hence, the chances of its children to survive are high as well. As a conclusion we have to adapt the invasive process in a way that every particle of the auxiliary process (of the non-invasive process) has the same probability to encounter an invasive particle. For this purpose we will not only start one invasive process but infinitely many.

In the following we describe a first approach that gives the right idea but does not lead to a good control of the marginal. First of all, there is a natural mapping from the family tree of the auxiliary process to the base graph: $\Psi : v \mapsto S_v$. Now, on the base graph we start infinitely many independent BRWs according to $(p_{\mathbf{i}}, q_{\mathbf{i}})$ as follows. Let $N \in \mathbb{N}$ (to be chosen later) and

start independent copies of invasive BRWs on each x with $|\Psi^{-1}(x)| = N$. Using these BRWs we define a random labeling of the base graph G: a vertex is labeled $\dagger$ if it is visited by some invasive particle at some time and $\bullet$ otherwise. In [7] it was shown that the trace of a (weakly surviving) BRW is a URG and moreover that the above labeling defines a URN. We use now the map Ψ to retrieve this labeling; label a vertex $v \in \mathbb{T}^{aux}$ with $\dagger$ if $\Psi(v)$ is labeled by $\dagger$ and label it with $\bullet$ otherwise. Each of the above steps is invariant under re-rooting and so is the new labeled version of $\mathbb{T}^{aux}$. Finally, due to Lemma 2.3 the connected component of $\mathbb{T}^{aux}$ with respect to the percolation induced by $\dagger$ is a URT. What remains therefore to proof is that the non-invasive BRW survives with positive probability when being confronted with an infinity of invasive BRWs. This would imply coexistence of the two original processes, since coexistence does not depend on the starting position of the processes.

4.2 Control of the marginal

In general it is not possible to control the marginal of the above invariant percolation. In fact, we need a better control of the "number" of invasive processes. Denote by $\mathcal{B}(n,o) = \{x: \ d(o,x) \leq n\}$ the ball of radius n around the origin o and denote by $\mathcal{S}(o,n) = \{x: \ d(o,x) = n\}$ the corresponding sphere. The growth rate g of the base graph G is defined as $g = \lim_{n\to\infty} \frac{1}{n} \log(|\mathcal{B}(n,o)|)$.

As the underlying random walks are supposed to be symmetric random walks we have, see [26, Lemma 8.1], that $p^{(n)}(x,y) \leq \rho^n$ for all $x, y \in G$ and all $n \in \mathbb{N}$. Two consequences of this fact on BRWs are given in the following lemma.

Lemma 4.1. *Let (μ, P) be a weakly surviving BRW on a non-amenable Cayley graph. Then*

1) $G(x,y|m) := \sum_{n=0}^{\infty} p^{(n)}(x,y) m^n \leq (m\rho)^{d(x,y)}/(1-\rho m)$;

2) there exists some constant ℓ such that $\limsup_{n\to\infty} \mathbb{E}[|\{v : S_v \in \mathcal{B}(o,n)\}|]/m^{\ell n} = 0$.

Proof. 1) Since the random walk is nearest neighbor, *i.e., $supp(q) = S$*, we have that

$$G(x,y|m) = \sum_{n=d(x,y)}^{\infty} p^{(n)}(x,y) m^n \leq \sum_{n=0}^{\infty} (\rho m)^{n+d(x,y)} \leq (m\rho)^{d(x,y)} \frac{1}{1-\rho m}.$$

2) Denote $R_n = \inf\{k \geq 0: \ S_v \notin \mathcal{B}(o,n) \ \forall |v| \geq k\}$. In the following denote by C a constant that is always chosen sufficiently large and may change from formula to formula. For some $b > 0$ (to be chosen in a moment) we obtain using the Markov inequality

$$\begin{aligned} \mathbb{P}(R_n > bn) &= \mathbb{P}(\exists v: \ |v| > bn: \ S_v \in \mathcal{B}(o,n)) \\ &\leq \sum_{k>bn} \sum_{y \in \mathcal{B}(o,n)} m^k p^{(k)}(o,y) \\ &\leq C \sum_{k>bn} m^k \rho^k g^n \leq C(g(m\rho)^b)^n. \end{aligned}$$

Hence, we can choose b sufficiently large such that the latter probability is summable. Eventually, the Lemma of Borel-Cantelli assures that $\limsup R_n/n \leq b$. Finally, for $\ell > 0$

$$\limsup_{n\to\infty} \frac{1}{m^{\ell n}} \mathbb{E}[|\{v : S_v \in \mathcal{B}(o,n)\}|] \quad \leq \quad \limsup_{n\to\infty} \frac{1}{m^{\ell n}} \mathbb{E}[|\{v: \ |v| \leq R_n\}|]$$

$$
\begin{aligned}
&\leq \mathbb{E}[\limsup_{n\to\infty} \frac{1}{m^{\ell n}} |\{v: \ |v| \leq R_n\}|] \\
&\leq \mathbb{E}[\limsup_{n\to\infty} \frac{1}{m^{\ell n}} |\{v: \ |v| \leq (b+1)n\}|] \leq \frac{d^{(b+1)n}}{m^{\ell n}}
\end{aligned}
$$

and we can choose ℓ sufficiently large such that the latter term goes to 0. □

The first part of Lemma 4.1 is used to control each of the invasive processes and the second part to adjust the "number" of these invasive processes. In order to start with the adjustment let us consider a non-invasive process with less branching. For any constant $\gamma \in (0,1]$, to be chosen later, we define the truncated Galton-Watson process by

$$
\mu_k^{(\gamma)} = \begin{cases} \gamma \mu_k^{\mathbf{n}}, & \text{for } k \geq 2, \\ \mu_1^{\mathbf{n}} + (1-\gamma)\sum_{k=2}^{\infty} \gamma \mu_k^{\mathbf{n}}, & \text{for } k = 1, \\ \mu_0^{\mathbf{n}}, & \text{for } k = 0. \end{cases}
$$

and denote its mean by m_γ. This construction is made to ensure two main properties: $m_\gamma \to 1 - \mu_0^{\mathbf{n}} \leq 1$ as $\gamma \to 0$ and $\mu_k^{(\gamma_1)} < \mu_k^{(\gamma_2)}$ for all $\gamma_1 < \gamma_2$ and $k \geq 2$. This latter property allows to construct a natural coupling of the original and the "γ-processes". Hence, denote by S^γ the BRW corresponding to the family tree $\mathbb{T}^\gamma$. Due to the coupling, it remains to show that the "γ-process" has positive probability of survival for some $\gamma > 0$.

Recall the definition of Ψ, see Subsection 4.1, and start independent copies of BRW on each x with $|\Psi^{-1}(x)| = N$. (The constant N is still to be chosen.) We will also denote by $\mathbb{P}$ the probability measure describing the non-invasive process together with the infinite number of invasive processes. Denote by $A \subset G$ the (random) set where invasive processes are started. For $x \in A$ we denote by $S_v^{\mathbf{i},x}$ the invasive BRW started in x with family tree $\mathbb{T}^{\mathbf{i},x}$. We have

$$
\sum_{x \in \mathcal{S}(o,n)} \mathbb{P}(x \in A) \leq \mathbb{E}[|\{v: \ S_v^\gamma \in \mathcal{S}(o,n), \xi(v) = \dagger\}|],
$$

here $\xi(v)$ denotes the mark of the vertex v induced by Ψ. Due to Lemma 4.1 for any $\gamma \in (0,1)$ there exists some constants C_γ and ℓ_γ such that $\mathbb{E}[|\{v: \ S_v^\gamma \in \mathcal{S}(o,n)\}|] \leq C_\gamma m_\gamma^{\ell_\gamma n}$. Since the trace of the BRW is unimodular we have that there exists a constant $C_N \to 0$ (as $N \to \infty$) such that

$$
\mathbb{E}[|\{v: \ S_v^\gamma \in \mathcal{S}(o,n), \xi(v) = \dagger\}|] \leq C_N C_\gamma m_\gamma^{\ell_\gamma n}.
$$

Moreover, the proof of Lemma 4.1 gives that the constant ℓ_γ can be chosen uniform in γ since there is a natural coupling for the last exit times R_n of different "γ-processes". Hence, there exists some constant ℓ such that for all $\gamma \in (0,1)$

$$
\sum_{x \in \mathcal{S}(o,n)} \mathbb{P}(x \in A) \leq C_N C_\gamma m_\gamma^{\ell n}.
$$

Using this together with a union bound and part 1) of Lemma 4.1 we obtain

$$
\mathbb{P}(\xi(\mathbf{r}) = \dagger) \quad \leq \quad \mathbb{P}\left(\exists x \in A, \exists v \in \mathbb{T}^{\mathbf{i},x} : S_v^{\mathbf{i},x} = S_r^\gamma\right)
$$

$$
\begin{aligned}
&\le \sum_{x\in G} \mathbb{P}\left(x\in A, \exists v\in \mathbb{T}^{\mathbf{i},x} : S_v^{\mathbf{i},x} = S_r^{\gamma}\right)\\
&= \sum_{n=0}^{\infty} \sum_{x\in\mathcal{S}(o,n)} \mathbb{E}\left[|\{v:\ S_v^{\mathbf{i},x} = S_{\mathbf{r}}^{\gamma}\}|\mid x\in A\right]\mathbb{P}(x\in A)\\
&\le \sum_{n=0}^{\infty} \frac{1}{1-\rho_{\mathbf{i}} m_{\mathbf{i}}} (m_{\mathbf{i}}\rho_{\mathbf{i}})^n \sum_{x\in\mathcal{S}(o,n)} \mathbb{P}(x\in A)\\
&\le \sum_{n=0}^{\infty} \frac{1}{1-\rho_{\mathbf{i}} m_{\mathbf{i}}} (m_{\mathbf{i}}\rho_{\mathbf{i}})^n C_N C_\gamma m_\gamma^{\ell n}.
\end{aligned}
$$

We can choose γ sufficiently small such that $m_\gamma^{\ell} m_{\mathbf{i}}\rho_{\mathbf{i}} < 1$. Let $c_{\mathbf{UGW}_\gamma}$ be the constant from Theorem 2.4 for the Galton-Watson with offspring distribution $\mu^{(\gamma)}$. Now, choose N sufficiently large (which makes C_N sufficiently small) such that the marginal $\mathbb{P}(\Xi(\mathbf{r}) \neq \dagger) > c_{\mathbf{UGW}_\gamma}$. This in turn implies that the non-invasive BRW with offspring distribution $\mu^{(\gamma)}$ survives with positive probability if confronted with an infinite number of invasive BRWs. Hence, for some $\gamma_c \le 1$ there is coexistence of one invasive and one non-invasive BRW since coexistence does not depend on the choice of the starting positions of the processes. Eventually, using the monotonicity in γ a standard coupling argument implies coexistence for all $\gamma \in [\gamma_c, 1]$.

Acknowledgment

The idea behind this paper came up during the focussed meeting “Branching Random Walks and Related Topics” in Graz. The author thanks TU-Graz and RGLIS for organization and funding. Moreover, the author thanks the Department of Mathematical Structure Theory of TU-Graz for their hospitality during the write up. The author thanks Itai Benjamini for helpful comments on a first version of this note.

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